TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 285, Number 2, October 1984

SURGERY ON POINCARÉ COMPLEXES

BY

J. P. E. HODGSON

ABSTRACT. A geometric approach to surgery on Poincaré complexes is described. The procedure mimics the original techniques for manifolds. It is shown that the obstructions to surgering a degree-one normal map of Poincaré complexes to a homotopy equivalence lie in the Wall groups, and that all elements in these groups can arise as obstructions.

0. Introduction. In this paper we describe a technique for surgery in the Poincaré duality category which extends the usual geometric description of surgery on manifolds given by Wall in his book [**10**]. The main result which overlaps with the results of Jones [**4**] and Quinn [**7**] is that the obstruction groups of the Poincaré duality category are the same as for the manifold case.

Suppose that we have a degree-one normal map $f\colon P \to X$ between two Poincaré duality spaces of formal dimension n; this means, in particular, that f lifts to a map of the Spivak normal bundles. The heart of our technique lies in showing that this normal bundle information allows one to perform surgeries below the middle dimension and then to represent the obstructions to completing the surgery inside a Poincaré embedded subcomplex of P which has a smooth structure. Using this smooth structure one can define the surgery obstructions in a way that is precisely parallel to the definitions given by Wall. More specifically, every Poincaré duality space of formal dimension $\geqslant 6$ has a smooth structure on the neighborhood of its 2-skeleton. Once one has done surgery to confine the kernels to the middle dimensions, it is possible to represent generators for these kernels inside a smooth region of the Poincaré complex, which is obtained by adding handles to the smooth 2-skeleton.

This paper has the following organization. §1 contains the definitions and notations that will be used. §2 contains some technical results on Poincaré complexes and the central lemma which makes everything work and which we call the Surgery Preparation Lemma. The remaining paragraphs unwind the details, so that §3 contains the proof of the $(\pi\text{-}\pi)$ theorem for Poincaré duality spaces of formal dimension $\geqslant 7$ (Wall's "important special case" [**9**]). §4 describes the obstructions to surgery in the middle dimension for Poincaré spaces of formal dimension $\geqslant 6$. Finally, in §5 we show how the techniques can be improved to yield the $(\pi\text{-}\pi)$ theorem in dimension 6 and to give surgery obstructions in dimension 5.

Received by the editors January 31, 1983 and, in revised form, November 22, 1983.
1980 *Mathematics Subject Classification*. Primary 57P10, 57R65; Secondary 57Q65.
Key words and phrases. Poincaré duality space, normal map, surgery obstruction.

In one sense this paper is an extended exercise based on the general position results of [**3**]; however, it is presented separately, partly because of the intrinsic interest of the result and partly because it has proved possible to improve the argument foreshadowed in [**3**] to yield results in the dimension 5, 6 and 7. An earlier version of this paper appeared in preprint form in 1973, at which time the dimension 5, 6 and 7 results were absent. I should like to thank Bill Browder for encouraging me to rework the paper. I would also like to thank Larry Siebenmann for his helpful comments.

1. Normal maps of Poincaré complexes and surgery below the middle dimension. The contents of this section are by now well known; however, it seems appropriate to gather together the definitions and notations with which we shall be working. Since our approach to the problem grew out of our efforts to understand Levitt's embedding theory [**5**] for Poincaré duality spaces, our definitions reflect this geometric bias.

DEFINITION 1.1. A Poincaré complex of formal dimension n is a finite connected CW-complex X, such that if N is a regular neighborhood of X in some embedding j: $X \subset S^{n+q}$, q large, then the map $\partial N \to N$ is homotopy equivalent to a spherical fibration with fibre S^{q-1}.

A Poincaré pair of formal dimension n is a finite CW-pair (X, Y) with X connected, such that if (M, N) is a regular neighborhood pair of a proper embedding j: $(X, Y) \hookrightarrow (D^{n+q}, S^{n+q-1})$, q large, then the map $\partial M - \operatorname{Int} N \hookrightarrow M$ is homotopy equivalent to a spherical fibration with fibre S^{q-1}.

We recall the notion of an embedding in the Poincaré duality category. For the sake of brevity we give only the definition for maps of pairs.

DEFINITION 1.2. Given a Poincaré pair (P, Q) of formal dimension n and a finite CW-pair (K, L) of dimension (k, l), a map of pairs f: $(K, L) \to (P, Q)$ is said to be homotopic to a Poincaré embedding if there is a splitting of P mod boundary which realizes f as one 'half' of the splitting.

More precisely, P can be written as a union $P = (N_1, \partial N_1) \cup_{N_0} (N_2, \partial N_2)$ with Q split as $(\partial N_1 - \operatorname{Int} N_0) \cup_{\partial N_0} (\partial N_2 - \operatorname{Int} N_0)$, and there is a homotopy equivalence of pairs

$$k: (K, L) \to (N_1, \partial N_1 - \operatorname{Int} N_0)$$

such that the following diagram commutes up to homotopy:

$$\begin{array}{ccc} (K, L) & \to & (P, Q) \\ \downarrow k & & \wr\| \\ (N_1, \partial N_1 - \operatorname{Int} N_0) & \hookrightarrow & (N_1 \cup N_2, \partial N_1 \cup \partial N_2) \end{array}$$

(See Diagram 1.) Furthermore, the inclusions $N_0 \hookrightarrow N_1$ and $\partial N_1 - \operatorname{Int} N_0 \hookrightarrow N_1$ are $(n - k - 1)$-connected and $(n - l - 2)$-connected, respectively.

The spherical fibration which appears in Definition 1.1 is called the Spivak normal bundle [**8**] of the Poincaré complex (respectively Poincaré pair). A normal map of Poincaré complexes is required to respect the Spivak normal bundle.

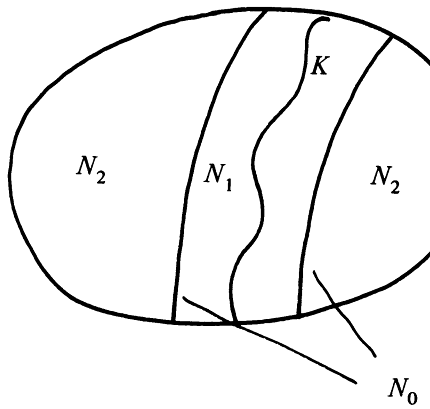

DIAGRAM 1

DEFINITION 1.3. Let X, X' be Poincaré complexes of formal dimension n, with Spivak normal bundles ν_X and ν_X', respectively. A commutative diagram

$$\begin{array}{ccc} \nu_X & \xrightarrow{\bar f} & \nu_{X'} \\ \pi \downarrow & & \downarrow \pi' \\ X & \xrightarrow{f} & X' \end{array}$$

where f is a bundle map covering f and preserving Thom classes is called a degree-one map of Poincaré complexes. A similar definition applies to Poincaré pairs.

The following result is due to Levitt [**6**].

THEOREM 1.4. *Let P^n be a finite connected CW-complex and f: $P \to X$ a map. Suppose that there is a spherical fibration ξ over X and a stable equivalence of the pull-back $f^*(\xi)$ with the Spivak normal bundle ν of P. Then f can be made k-connected by a finite sequence of (Poincaré) surgeries, provided $n \geqslant 2k$ and $n \geqslant 5$.*

Levitt proved this result using the embedding theorems he had proved in [**5**] to carry over the usual surgery arguments of Wall [**9**]. However, these embedding theorems are not strong enough to give us the results we need in the middle dimension. We use ideas drawn from our earlier paper [**3**] to circumvent these difficulties.

2. The Surgery Preparation Lemma. We approach the problem of defining surgery obstructions for a degree-one normal map f: $P \to X$ of Poincaré complexes in the tradition of the earliest exposition of surgery. That is to say we suppose that we have done all the surgeries allowed us by Theorem 1.4 and are left with kernels in the middle dimension. We then show that we can represent the Wall kernels by maps of spheres into a codimension zero subcomplex of P which has a smooth structure. This associates to our original surgery problem a smooth surgery problem to which we can attach a Wall obstruction in the usual way. Finally, it will be incumbent upon us to show that these obstructions are well defined invariants of normal cobordism classes.

In order to do this down through the full-dimensional range that we are seeking, we need the following improvement of the main result of [3]. (If one relies only on the result of [3] then we have to impose the condition that all our Poincaré complexes have formal dimension at least 9.)

LEMMA 2.1 (GENERAL POSITION LEMMA). *Let (X, Y) be a finite Poincaré pair of formal dimension n and suppose we are given a map*

$$f\colon (D^k, S^{k-1}) \to (X, Y)$$

where $k \leqslant n - 3$. Then there is a CW-pair (K, L) obtained from (D^k, S^{k-1}) by adding cells of dimensions $(2k - n + 2, 2k - n + 1)$ and a map $g\colon (K, L) \to (X, Y)$ homotopic to a Poincaré embedding, such that the following diagram commutes up to homotopy:

$$\begin{array}{ccc} (D^k, S^{k-1}) & \xrightarrow{f} & (X, Y) \\ \searrow & & \nearrow g \\ & (K, L) & \end{array}$$

PROOF. The strengthening that this result represents over the main result of [3] is that we are able to drop the metastability condition $2n \geqslant 3k + 4$. The reasoning that allows us to do this is as follows. In [3] the condition $2n \geqslant 3k + 4$ was invoked at various stages in the proof to show that the modifications being made to the intermediate pairs (K_i, L_1) did not introduce any new singularities whose dimensions exceeded that below which we were trying to reduce the dimension of the singularity set. In fact, this represents a very demanding way to avoid trouble; it is sufficient to require that at each step any new singularities that may be introduced have dimension strictly smaller than the dimensions of the singularity with which one is currently concerned. This is achieved by the condition $n - k \geqslant 3$, provided one is careful to note that the singularity set is stratified by the multiplicities of the points in the singularity set and cells are added according to the codimension of the cells in the strata to which they belong. In fact, in [3] we had only to deal with double points and the cells we introduced corresponded to double points; now we have also to deal with triple points and so on.

We next introduce a notion which will be extremely useful in what follows.

DEFINITION 2.2. A Poincaré complex P of formal dimension n is said to have a smooth k-skeleton if there exists a splitting of $P = N_1 \cup N_2$, for which $(N_1, \partial N_1)$ can be given a smooth structure, and for which the map $f\colon N_1 \to P$ is k-connected.

With this definition in hand we state the following

PROPOSITION 2.3. *A Poincaré complex of formal dimension $\geqslant 4$ has a smooth 1-skeleton; if the formal dimension is greater than or equal to 6, it has a smooth 2-skeleton.*

PROOF. We consider first the case of formal dimension $\geqslant 6$. There is a CW-complex K of dimension $\leqslant 2$ and a map $f\colon K \to P$ which is 2-connected. By Levitt's embedding theorem [**5**], f is homotopic to a Poincaré embedding. Thus P splits as $N_1 \cup N_2$ with N_1 homotopy equivalent to K and the inclusion $N_1 \to P$ being 2-connected. Now the Spivak normal bundle of N_1 is classified by a map $N_1 \to BG$ (the classifying space for spherical fibrations) and this map lifts to BO, since $\pi_1(G/O) = \pi_0(G/O) = 0$. Hence, using smooth surgery theory (specifically Theorem 3.3 of [**9**]) $(N_1, \partial N_1)$ is smoothable.

In the case of formal dimension $\geqslant 4$, but less than 6, one embeds a 1-complex K with corresponding splitting $P \simeq N_1 \cup N_2$, and $N_1 \sim K$. One cannot now use surgery to smooth N_1; however, Levitt's embedding theorem does permit one to assume that N_1 has a Poincaré handle decomposition modelled on K and the vanishing of $\pi_0(G/O)$ implies that N_1 can then be smoothed.

COMPLEMENT 2.4. *For fixed choice of K, the smooth 2-skeleton is unique if $n \geqslant 6$.*

PROOF. $(N_1, \partial N_1)$ is a stable smooth thickening of a 2-complex and so is classified by an element of $[K, BO]$ using Wall [**10**] and in this range $[K, BO] \simeq [K, BG]$.

We now come to the main results of this section; before giving the formal statement of the result we give an informal description of it.

If $f\colon P \to X$ is a degree-one map of Poincaré complexes, then any element of a Wall kernel is represented by a map $g\colon S^i \to P$ such that $f \circ g$ maps into the 2-skeleton of X. We need to know that when we apply the general position result 2.1 to g the resulting map $g'\colon K \to P$ is still such that $f \circ g'$ maps to the 2-skeleton, so that the Poincaré neighborhood of K in P can be smoothed since its normal bundle pulls back from a spherical bundle after a complex of dimension $\leqslant 2$.

LEMMA 2.5 (SURGERY PREPARATION LEMMA). (FIRST FORM) (a) *Let $g\colon P \to X$ be a degree-one map of Poincaré complexes of formal dimension $n \geqslant 6$. Let L be a finite CW-complex of dimension $\leqslant (n+1)/2$ and $f\colon L \to P$ a map such that the composition $g \circ f\colon L \to P$ is homotopic to a map into the 2-skeleton of X. Then there exists a CW-complex K obtained from L by adding cells of dimension $\leqslant 3$, and a Poincaré embedding $j\colon K \to N_1 \subset (N_1 \cup N_2) \simeq P$ and such that the diagram*

$$\begin{array}{ccc} L & \xrightarrow{f} & P \\ & \searrow \quad \nearrow j & \\ & K & \end{array}$$

commutes up to homotopy. Furthermore, $g \circ j\colon K \to X$ is homotopic to a map into the 2-skeleton $X^{(2)}$ of X and N_1 has a smooth structure.

(b) (*Relative Version*). *Let $g\colon (P, Q) \to (X, Y)$ be a degree-one normal map between Poincaré pairs of formal dimension $n \geqslant 7$. Suppose (L, L_0) is a finite CW-pair with* $\dim L \leqslant (n+1)/2$ *and* $\dim(L_0) \leqslant \dim L - 1$, *let $f\colon (L, L_0) \to (P, Q)$ be a map such that the composition $g \circ f\colon (L, L_0) \to (X, Y)$ is homotopic as a map of pairs to a map into the 2-skeleton pair $(X^{(2)}, Y^{(2)})$ of (X, Y), and further that f/L_0 is homotopic to a Poincaré embedding. Then there is a CW-pair (K, L_0) obtained by* *adding cells of dimension* $\leqslant 3$ *to* $(L - L_0)$ *and a Poincaré embedding*

$$j: (K, L_0) \to (N_1, \partial N_1) \hookrightarrow (N_1, \partial N_1) \cup_{N_0} (N_2, \partial N_2) \simeq (P, Q)$$

such that the diagram

$$\begin{array}{ccc} (L, L_0) & \xrightarrow{f} & (P, Q) \\ \searrow & & \nearrow j \\ & (K, L_0) & \end{array}$$

commutes up to homotopy. Furthermore, $g \circ j$ *is homotopic as a map of pairs to a map into the* 2-*skeleton pair of* (X, Y) *and* $(N_1, \partial N_1)$ *has a smooth structure with* $(\partial N_1 - \operatorname{Int} N_0)$ *a smooth submanifold of* ∂N_1.

PROOF. (a) We need only consider the critical case $n = 2k + 1$, $\dim L = k + 1$ since in all other cases the added cells are of dimension $\leqslant 2$ and the result follows directly as in Proposition 2.3.

Let $X^{(2)}$ be a smooth 2-skeleton for X so that $X \simeq X^{(2)} \cup Y$. Using the Spivak normal bundle construction, we can thicken up P and X to PL-manifolds T and W, respectively. Since g is a normal map, we get a commutative diagram (up to homotopy):

$$\begin{array}{ccc} \partial T & \xrightarrow{\bar{g}} & \partial W \\ \downarrow & & \downarrow \\ T & \xrightarrow{g} & W \end{array}$$

Now we can split W so as to reflect the splitting of X as $X^{(2)} \cup Y$ so that $W = W^{(2)} \cup_{W_0} V$ with $W^{(2)} \sim X$ and $(\partial W^{(2)} - W_0) \to W^{(2)}$ homotopy equivalent to the restriction of the fibration $\partial W \hookrightarrow W$ to $W^{(2)}$.

By composing f with the inclusions $P \hookrightarrow T$ and $X \hookrightarrow W$ and using general position, we can suppose the following:

(i) $f: L \to T$ is a PL-embedding.

(ii) $g \circ f: L \to W$ is a PL-embedding with image in $W^{(2)}$.

Let us write ξ_T for the spherical fibration $\partial T \hookrightarrow T$ and ξ_W for the spherical fibration $\partial W \hookrightarrow W$. We have already noted that $\xi_{W^{(2)}} = \partial W^{(2)} - W_0 \hookrightarrow W^{(2)}$ is homotopy equivalent to $\xi_W|_{W^{(2)}}$. Since g is a normal map, we have a map

$$g_0: \xi_T | f(L) \to \xi_W | g \circ f(L) = \xi_{W^{(2)}} | g \circ f(L) \hookrightarrow \partial W^{(2)} - W_0.$$

Denote $\xi_T | f(L)$ by η and let M_η denote the mapping cylinder of this spherical fibration. We have a diagram

$$\begin{array}{ccccc} \eta & \to & \partial T & \to & \partial W \\ \downarrow & & \downarrow & & \downarrow \\ M_\eta & \xrightarrow{f} & T & \xrightarrow{g} & W \end{array}$$

and, in fact, $g \circ f(M_\eta) \hookrightarrow W^{(2)}$.

The left-hand half of this diagram corresponds to the stable model of the General Position Lemma [**3**, p. 313]. The General Position Lemma is proved by modifying L and M_η along with it. Examination shows that the cells added to L are adjoined using M_η as a guide and are added in a neighborhood of M_η. Since M_η maps into $W^{(2)}$ it follows that all the added cells of the General Position Lemma and the spherical fibration attached along with them also map to $W^{(2)}$ and $\partial W^{(2)} - \operatorname{Int} W_0$, respectively. Thus the embedded CW-complex K obtained from L by the General Position Lemma maps to $X^{(2)}$ and the normal bundle of its Poincaré neighborhood pulls back from the normal bundle of X restricted to $X^{(2)}$ and hence can be smoothed. (b) follows in an entirely similar way using collars on the boundaries.

We need one final lemma which will enable us to modify the fundamental groups of embedded subcomplexes of Poincaré complexes, so that they are the same as that of the ambient complex.

LEMMA 2.6. (a) *Let P be a finite CW-complex which is a Poincaré complex of formal dimension $n \geqslant 6$. Suppose further that P is split as $M_1 \cup M_2$, with M_1 homotopy equivalent to a finite complex of dimension k, $k + 3 \leqslant n$, and $\pi_1(\partial M_1) \cong \pi_1(M)$ via the inclusion. Then we can exchange* 1- *and* 2-*handles between M_1 and M_2 to make the maps $\pi_1(\partial M_1) \to \pi_1(M_1) \to \pi_1(P)$ induced by the inclusion isomorphisms.*

(b) *Let $(P, \partial P)$ be a finite Poincaré pair of formal dimension $n \geqslant 7$, and suppose that $(P, \partial P)$ is split as $(P, \partial P) \simeq (M_1, \partial M_1) \cup_{M_0} (M_2, \partial M_2)$ so that*

$$\partial P \simeq (\partial M_1 - \operatorname{Int} M_0) \cup_{\partial M_0} (M_2 - \operatorname{Int} M_0).$$

Suppose that the inclusions $\partial M_0 \hookrightarrow M_1 - \operatorname{Int} M_0$ and $M_0 \hookrightarrow M_1$ induce isomorphisms on π_1, and that $(M_1, \partial M_1 - \operatorname{Int} M_0)$ is homotopy equivalent to a finite CW-pair (K, L) with $(\dim K) + 3 \leqslant n$ and $\dim L \leqslant \dim K - 1$. Then we can exchange handles between M_1 and M_2 so that the homomorphisms

$$\pi_1(\partial M_0) \to \pi_1(\partial M_1 - \operatorname{Int} M_0) \to \pi_1(\partial P)$$

and

$$\pi_1(M_0) \to \pi_1(M_1) \to \pi_1(P)$$

are isomorphisms.

PROOF. (a) Let $\pi_1(P)$ have a presentation $\pi_1(P) = \{a_1, \ldots, a_s; r_1, \ldots, r_t\}$ so that each r_i is a word in $a_1, \ldots, a_s$ and $\pi_1(P)$ is the quotient of the free group generated by $a_1, \ldots, a_s$ by the smallest normal subgroup containing $(r_1, \ldots, r_t)$. Similarly, we will have $\pi_1(M_1) = \{x_1, \ldots, x_k; y_1, \ldots, y_l\}$.

Since the formal dimension of M_1 exceeds the homotopy dimension by at least 3, we have $\pi_1(\partial M_1) \simeq \pi_1(M_1)$, so that we can use duality and excision to represent the generators a_i of $\pi_1(P)$ by maps $\alpha_i\colon S^1 \to M_2$, which can then be deformed to maps,

$$\alpha_i\colon (I, \{0,1\}) \to (M_2, \partial M_2) \hookrightarrow (P, M_1)$$

By Levitt's embedding theorem the α_i can be supposed to be disjoint Poincaré embeddings. We can thus add the neighborhoods of these 1-cells to M_1 and subtract them from M_2. (See Diagram 2.)

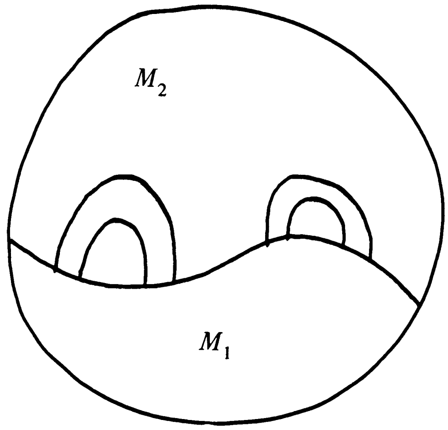

DIAGRAM 2

M_1 becomes $M_1' = M_1 \cup$ 1-handles and M_2 becomes $M_2' -$ (1-handles) and we have $\pi_1(M_1') = \pi_1(M_1) * F$ where F is a free group with generators $g_1, \ldots, g_s$ corresponding to the generators $a_1, \ldots, a_s$ of $\pi_1(P)$. Thus

$$\pi_1(M_1') = \{x_1, \ldots, x_k, g_1, \ldots, g_s; y_1, \ldots, y_l\}.$$

The map $i_*: \pi_1(M_1') \to \pi_1(P)$ is onto by construction, and Lemma IV.1.15 on p. 92 of Browder's book [**1**] shows that the kernel of i_* is the smallest normal subgroup containing a specified finite set of elements of $\pi_1(M_1')$. Since $\dim(\partial M_1') \geq 4$ the elements can be represented by disjoint circles embedded in $\partial M_1'$. Moreover, these circles are null-homotopic in P and since $(\dim P - \text{homotopy}\dim(M_1')) \geq 3$ the null homotopy can be realized in M_2'. Thus we can use Levitt's embedding theorem in M_2' to realize the null-homotopy by Poincaré embedded 2-discs. These 2-discs are then the cores of 2-handles which can be exchanged between M_1' and M_2' to obtain $P \simeq M_1'' \cup M_2''$ with $\pi_1(\partial M_1'') \simeq \pi_1(M_1'') \simeq \pi_1(P)$ as required.

Note that by construction M_1 can be Poincaré embedded in M_1''. (b) of the theorem follows by working first on the boundary and then using collars fixing up the interior. As a corollary, we obtain the strong form of the Surgery Preparation Lemma.

LEMMA 2.7 (SURGERY PREPARATION LEMMA). (STRONG FORM) (a) *Under the hypotheses of Lemma* 2.5(a), *we can conclude that L may be modified to K by adding cells of dimension* ≤ 3 *so that the Poincaré neighborhood N of K in P has* $\pi_1(\partial N) \simeq \pi_1(N) \simeq \pi_1(P)$.

(b) *Similarly in the relative case.*

PROOF. The proof of Lemma 2.6 shows how to add additional 1- and 2-cells to the K produced by Lemma 2.5 to obtain the desired isomorphisms of the fundamental groups. The fact that K is initially Poincaré embedded in codimension ≥ 3 ensures that the fundamental group hypotheses of Lemma 2.6 are satisfied.

3. The (π-π) theorem for Poincaré pairs. In this paragraph we prove Wall's "Important Special Case" [**9**] for Poincaré pairs. Our method follows Wall's original argument closely so we shall refer to his treatment frequently. In particular, the algebraic portion of his argument will not be reproduced here except as required for clarity of presentation. This leaves us free to show how the Surgery Preparation Lemma permits us to translate Wall's geometric arguments into the Poincaré category. The result is as follows.

THEOREM 3.1. *Let* $(X, Y_0 \cup Y_1)$ *be a finite Poincaré pair of formal dimension* $n \geqslant 7$ *with* Y_0, Y_1 *Poincaré complexes of formal* $\dim n - 1$, *such that the inclusion* $Y_1 \hookrightarrow X$ *induces an isomorphism of the fundamental groups. Let* $f\colon (P, Q_0 \cup Q_1) \to (X, Y_0 \cup Y_1)$ *be a degree-one normal map of Poincaré pairs. Suppose further that* f *induces a homotopy equivalence* $f\colon Q_0 \to Y_0$. *Then we can perform Poincaré surgery on* P (*away from* Q_0) *to make* f *a homotopy equivalence.*

PROOF. Since Q_0 and Y_0, in fact, play no role in the argument, it will be enough to consider the case where they are both empty. Hence we can suppose that we have a degree-one normal map $f\colon (P, Q) \to (X, Y)$ of Poincaré pairs of formal dimension n, and that the inclusion $i\colon Y \hookrightarrow X$ induces an isomorphism of fundamental groups.

As usual there are two cases to consider, given by the parity of n.

Case 1. $n = \dim[X]$ even $= 2k$.

By Theorem 1.4, as in Wall [9], we can suppose that we have done surgery so that $f\colon Q \to Y$ is $(k-1)$-connected and $f\colon P \to X$ is k-connected.

Since $k \geqslant 3$ all four fundamental groups are isomorphic with the isomorphisms induced by the obvious maps. We, therefore, have the following sequence of isomorphisms, where Λ denotes the group ring $Z[\pi_1(x)]$, $\tilde{f}$ the map of universal covers induced by f, and $K_k(P, Q)$ the Wall kernel:

$$\pi_{k+1}(f) \cong \pi_{k+1}(\tilde{f}) \cong H_{k+1}(\tilde{f}) = H_{k+1}(f; \Lambda) = K_k(P, Q).$$

As in [9], by performing surgeries on trivial $(k-1)$-spheres in Q we can make $K_k(P, Q)$ free and based. Let (e_i), $i = 1, \ldots, l$, denote such a preferred base. Corresponding to each e_i we have a map α_i

$$\begin{array}{ccccccc} D^k_+ & \hookrightarrow & D^{k+1} & & P & \xrightarrow{f} & X \\ \uparrow & & \uparrow & \xrightarrow{\alpha_i} & \uparrow & & \uparrow \\ S^{k-1} & \hookrightarrow & D^k_- & & Q & \to & Y \end{array}$$

representing e_i as an element of $\pi_{k+1}(f)$. Restricting to (D^k_+, S^{k-1}) gives us a collection of maps

$$\alpha_i\colon \left(D^k_+, S^{k-1}\right) \to (P, Q), \qquad i = 1, \ldots, l.$$

We can wedge these maps together to obtain a single map

$$A\colon \left(\bigvee_{i-1}^{l} D^i, \bigvee_{i=1}^{l} S^{k-1} \right) \to (P, Q).$$

We apply the strong form of the Surgery Preparation Lemma, applying part (a) to $A|\bigvee_{i=1}^{l} S^{k-1}$ first and then the relative version. If we do this we obtain a finite CW-pair (K, L) where $\dim(L - \bigvee_{i=1}^{l} S^{k-1}) \leqslant 2$ and $\dim(K - \bigvee_{i=1}^{l} D^{k}) \leqslant 2$, together with a splitting of P mod boundary $(P, Q) = (M_1, \partial M_1) \cup_{M_0} (M_2, \partial M_2)$ such that $(M_1, \partial M_1 - \operatorname{Int} M_0)$ is homotopy equivalent to (K, L) and has a smooth structure. Furthermore, since we have invoked the strong form we have isomorphisms of fundamental groups,

$$\pi_1(\partial M_0) \simeq \pi_1(\partial M_1 - \operatorname{Int} M_0) \simeq \pi_1(Q) \cong \pi_1(P),$$
$$\pi_1(M_0) \simeq \pi_1(M_1) = \pi_1(P).$$

Finally, there is a commutative diagram:

$$\begin{array}{ccc}
\left(\bigvee_{i=1}^{l} D^{k}, \bigvee_{i=1}^{l} S^{k-1}\right) & \to & (P, Q) \\
\downarrow & \searrow & \uparrow \\
(K, L) & \to & (M_1, \partial M_1 - \operatorname{Int} M_0)
\end{array}$$

The generators of the kernel $K_k(P, Q)$ can thus be represented by maps that factor through $(M_1, \partial M_1 - \operatorname{Int} M_0)$. We can, therefore, use the argument of p. 40 in Wall **[9]** to obtain disjoint framed embeddings in the smoothed $(M_1, \partial M_1 - \operatorname{Int} M_0)$. We use these embeddings to perform handle subtraction and complete the proof using the algebraic arguments of Wall.

Case 2. $n = \dim[X]$ odd $= 2k + 1$.

In this case Theorem 1.4 allows us to perform surgeries so that we can suppose that f induces k-connected maps $f\colon P \to X$ and $f\colon Q \to Y$ and that $K_k(P, Q)$ is zero. Thus we have a short exact sequence of s-based $Z[\pi_1(x)]$-modules,

$$0 \to K_{k+1}(P, Q) \to K_k(Q) \to K_k(P) \to 0.$$

As before, we can perform surgeries on trivial $(k - 1)$-spheres in Q to convert all the s-bases into actual bases. Now $K_{k+1}(P, Q) \simeq \pi_{k+2}(f)$ so we can choose a basis $e_1, \ldots, e_l$ which is represented by maps α_i

$$\begin{array}{ccccccc}
D_{+}^{k+1} & \hookrightarrow & D^{k+2} & & P & \xrightarrow{f} & X \\
\uparrow & & \uparrow & \xrightarrow{\alpha_i} & \uparrow & & \uparrow \\
S^{k} & \hookrightarrow & D_{-}^{k+1} & & Q & \xrightarrow{f} & Y
\end{array}$$

and restricting to the D_{+}^{k+1}'s we obtain, as before, a map

$$A\colon \left(\bigvee_{i=1}^{l} D^{k+1}, \bigvee_{i=1}^{l} S^{k}\right) \to (P, Q).$$

We apply the General Position Lemma 2.1 to $A|\vee_{i=1}^{l} S^k$ to obtain a CW-complex L_0 by adding cells of dimension $\leqslant 2$ to $\vee_{i=1}^{l} S^k$ and a Poincaré embedding $j_0\colon L_0 \to Q$. The strong form of the Surgery Preparation Lemma now applies to the map A, to give a splitting $(P, Q) \simeq (M_1, \partial M_1) \cup_{M_0} (M_2, \partial M_2)$ such that $(M_1, \partial M_1 - \operatorname{Int} M_1)$ can be smoothed and $\pi_1(\partial M_0) \simeq \pi_1(\partial M_0 - \operatorname{Int} M_1) \simeq \pi_1(M_1) \simeq \pi_1(P)$ with the isomorphisms induced by inclusion. Furthermore, A factors through the inclusion of $(M_1, \partial M_1 - \operatorname{Int} M_0)$ into (P, Q). Since $(M_1, \partial M_1 - \operatorname{Int} M_0)$ is smoothable we can carry over Wall's argument on p. 41 of **[9]** to represent a preferred base for $K_{k+1}(P, Q)$ by framed immersions

$$f_i\colon (D^{k+1}, \partial D^{k+1}) \to (M_1, \partial M_1 - \operatorname{Int} M_0)$$

and these f_i can be modified by regular homotopies so that their boundaries define disjoint framed embeddings in $\partial M_1 - \operatorname{Int} M_0 \hookrightarrow Q$. We use these framed embeddings to add $(k + 1)$-handles to P along Q replacing the pair (P, Q) by a pair (P', Q') and getting a degree-one normal map $f'\colon (P', Q') \to (X, Y)$.

Now reading through Wall's argument on p. 42 of **[9]**, we note that the only nonalgebraic part of the argument is the proof that we can kill $K_k(P')$. To see that this can be done, choose a preferred base $h_1, \ldots, h_r$ for $K_k(P')$. These generators correspond to elements of $\pi_{k+1}(P')$ and wedging together representatives gives us a map $B\colon \vee_{i=1}^{r} S^k \to P'$. We again apply the Surgery Preparation Lemma to B (this time the weak form is sufficient) and get a CW-complex K by adding 1-cells to $\vee_{i=1}^{r} S^k$ and a splitting $P' = N_1 \cup_{\partial N_1} N_2$ with N_1 homotopy equivalent to K and having a smooth structure. Therefore, we can represent the basis for $K_k(P')$ by framed spheres embedded in N_1 and use smooth surgery to kill $K_k(P')$.

This completes the proof of Theorem 3.1.

COROLLARY 3.2. *Let $f\colon P \to X$ be a degree-one map of Poincaré complexes of formal dimension $\geqslant 6$, and suppose there is a normal cobordism*

$$F\colon (W; P, Q) \to (X; X \times 0, X \times 1)$$

such that $F|Q$ is a homotopy equivalence. Then one can do surgery on $f\colon P \to X$ to make f a homotopy equivalence.

PROOF. Theorem 3.1 allows one to do surgery keeping Q fixed to make F a homotopy equivalence of pairs. The restriction of the surgeries to P gives the desired surgeries on f.

4. The surgery obstructions. This paragraph is devoted to giving the definitions of the surgery obstructions for Poincaré surgery and showing that the definitions make sense. The idea is to use the Surgery Preparation Lemma to reduce the Poincaré surgery problems to smooth ones. We will restrict ourselves to the case without boundary; when there are boundaries, one can (for dimension $\geqslant 7$) proceed using the relative versions of the results in §2. Our main theorem is the following

THEOREM 4.1. *Let $f\colon P \to X$ be a degree-one normal map between Poincaré complexes of formal dimension $\geqslant 6$. Then we can associate to f an element $\Theta(f)$ which* *lies in the Wall obstruction group $L_n(\pi_1(X), w)$ where w is the "first Stiefel-Whitney class" of X. $\Theta(f)$ depends only on the normal cobordism class of f and $\Theta(f) = 0$ if and only if one can do surgery on f to obtain a homotopy equivalence.*

PROOF. We begin by choosing a smooth 2-skeleton $X^{(2)}$ for X, so that $X \simeq X^{(2)} \cup Y$. We note that the surgeries that we do on P below the middle dimension allow us to assume that P has a smooth 2-skeleton and that $P^{(2)} = X^{(2)}$.

THE DEFINITION OF $\Theta(f)$. By Theorem 1.4 we can suppose f has been surgered to be a k-connected map where k is determined by the relations $n = 2k$ or $2k + 1$, depending on the parity of n.

At this point f induces an isomorphism of fundamental groups and we want to kill $K_k(P)$. By performing surgeries on trivial spheres in P we can suppose that $K_k(P)$ is free and based. Let $e_1, \ldots, e_r$ be such a base for $K_k(P)$. Since $K_k(P) \simeq \pi_{k+1}(f)$ we can represent the elements k_i by maps α_i:

$$\begin{array}{ccc} (D^{k+1}, *) & \xrightarrow{\alpha_i} & (X, *) \\ \cup & & \cup \\ (S^k, *) & \xrightarrow{\alpha_i} & (P, *) \end{array}$$

At this point it is necessary to distinguish between the two cases $n = 2k$ and $n = 2k + 1$.

The case $n = 2k$. By wedging together the maps α_i we create a single map

$$A = \vee \alpha_i \colon \bigvee_{i=1}^{r} S^k \to P.$$

We apply the strong form of the Surgery Preparation Lemma to this map and get a CW-complex K obtained by adding cells of dimension $\leqslant 2$ to $\vee_{i=1}^{r} S^k$. Moreover, K is Poincaré embedded in P, so that P is split as $N_1 \cup N_2$ with $N_1 \simeq K$, and N_1 having a smooth structure. Finally, $\pi_1(\partial N_1) \simeq \pi_1(N_1) \simeq \pi_1(P)$ and the following diagram commutes up to homotopy:

$$\begin{array}{ccc} \bigvee_{i=1}^{r} S^k & \xrightarrow{A} & P \\ \downarrow & & \downarrow\wr \\ K & \hookrightarrow & N_1 \cup N_2 \end{array}$$

We, therefore, have a normal map $f\colon P \to X$ such that $f|N_1$ maps into $X^{(2)}$. Thus since the individual maps $\alpha_i\colon S^k \to P$ now factor through $N_1 \to X^{(2)}$, the generators of $K_k(P)$ can be represented by smoothly immersed framed k-spheres in N_1 as in the smooth case. An element $\Theta(f) \in L_n(\pi_1(X), w)$ can then be defined as in Wall **[9]** using the intersections of these immersions. We must show that $\Theta(f)$ is well defined.

Suppose first that we have obtained two splittings of P from the same set of generators $P \simeq M_1 \cup M_2$ and $P \simeq N_1 \cup N_2$. Then we have normal maps f_0: $P \to X$ with $f_0|M_1$ mapping to $X^{(2)}$ and f_1: $P \to X$ with $f_1|N_1$ mapping to $X^{(2)}$. We will construct a normal cobordism F: $P \times I \to X \times I$ which reduces to f_0 on $P \times 0$ and to f_1 on $P \times 1$. To do this we apply the relative version of the Surgery Preparation Lemma to the homotopy

$$B\colon \bigvee_{i=1}^{r} S^k \times I \to P \times I$$

which joins the maps A and A' giving the two different representations of the generators. By adding cells of dimension $\leqslant 3$ to $\vee_{i=1}^{r} S^k \times I$ we obtain a CW-complex K' which is Poincaré embedded in $P \times I$, whose Poincaré neighborhood W in $P \times I$ is smooth and meets $P \times 0$ in M_1 and $P \times 1$ in N_1. In W we can then shift the image of B to provide a regular homotopy between the two sets of immersions showing that $\Theta(f)$ is an invariant of the homotopy classes of the maps chosen to represent the base element $e_1, \ldots, e_r$.

We must also consider the effect of choosing a different basis for $K_k(P)$. It is clearly enough to consider a change of the form

$$e_1 \to e_1 + g \cdot e_2,\ e_2 \to e_2 \cdots e_r \to e_r$$

where $g \in \pi_1(P)$. This change can be introduced into our definition of $\Theta(f)$ in two ways, either at the beginning when the basis is chosen giving an obstruction Θ_1 or we can work with the old basis to find the smooth submanifold N_1 of P and then introduce the change giving an obstruction Θ_2. Clearly the result leads to homotopic representatives corresponding to the new basis so by the preceding remarks $\Theta_1 = \Theta_2$, and since Θ_2 is obtained from the original obstruction by a basis change inside the smooth manifold N_1, $\Theta_2 = \Theta(f)$. Thus Θ is well defined.

The case $n = 2k + 1$. We can again wedge together representatives for the generators of $K_k(P)$ to get a map A: $\vee_{i=1}^{r} S^k \to P$. We then apply the strong form of the Surgery Preparation Lemma to this map and get a CW-complex K which is $\vee_{i=1}^{r} S^k$ union cells of dimension $\leqslant 2$. P will be split as $N_1 \cup N_2$ with N_1 smooth, of the homotopy type of K and such that $\pi_1(\partial N_1) \simeq \pi_1(N_1) \simeq \pi_1(P)$. We therefore have a homotopy commutative diagram:

$$\begin{array}{ccc} \bigvee_{i=1}^{r} S^k & \longrightarrow & P \\ \downarrow & \overset{A'}{\searrow} & \uparrow \\ K & \longrightarrow & N_1 \end{array}$$

Since N_1 is smooth the diagonal map A': $\vee_{i=1}^{r} S^k \to N_1$ in the above diagram can be realized by an embedding. Let U be a smooth regular neighborhood of this embedding. We then have a map of triads

$$f\colon (P, P_0, U) \to (X, X_0, D^n)$$

where $P = P_0 \cup_{\partial U} U$ is a splitting of P and $X = X_0 \cup_{S^{n-1}} D^n$ is a splitting of X. This is exactly the information that Wall uses in [**9**, §6] to define the odd-dimensional obstructions $\Theta(f) \in L_n(\pi_1(X), W)$.

To see that $\Theta(f)$ is well defined we proceed much as in the even-dimensional case. We consider first the case where the same set of generators for $K_k(P)$ are represented by homotopic maps A_1, A_2: $\vee_{i=1}^r S^k \to P$, and let B: $\vee_{i=1}^r S^k \times I \to P \times I$ be a homotopy joining A_1 and A_2. If we apply the relative Surgery Preparation Lemma in the strong form to the map B, we obtain a splitting of $P \times I$ as $W_1 \cup W_2$ with W_1 smooth and with $(W_1 \cap P \times 0) \cup (W_2 \cap P \times 0)$ (resp. $(W_1 \cap P \times 1) \cup (W_2 \cap P \times 1)$) being the splittings corresponding to A_1 (resp. A_2) obtained from the Surgery Preparation Lemma. The image of B lies in W_1 and we can use Wall's arguments to show that the obstructions are the same for A_1 and A_2. A change of basis is treated as in the even-dimensional case.

$\Theta(f)$ *depends only on the normal cobordism class of* f. We have already seen in Corollary 3.2 that if f is normally cobordant to a homotopy equivalence then f can be surgered to a homotopy equivalence so that $\Theta = 0$. It remains to show that in general $\Theta(f)$ depends only on the normal cobordism class of f. As before we shall be adapting the arguments of Wall's book, the translations required are similar to those we have done before so we shall keep our comments brief.

In the odd-dimensional case $n = 2k + 1$, we follow Wall's Lemma 6.1 in [**9**]; thus, if F: $Q \to Y$ is a normal cobordism between k-connected maps f_0: $P_0 \to X \times 0$ and f_1: $P_1 \to X \times 1$ we can do surgery to make F' $(k + 1)$-connected. The proof of invariance reduces to showing that we can go from f_0 to f_1 by surgeries on k-spheres, since these surgeries do not change $\Theta(f)$. Thus we need to be able to represent a base for $K_{k+1}(Q, P_0)$ by handles attached to P_0; to do this we use the fact that $K_{k+1}(Q, P_0)$ is stably free and s-based so that we can perform surgeries on trivial k-spheres in Q to add free modules of rank 2 to $K_{k+1}(Q,P_0)$ until it becomes free and S-based. We then use the Surgery Preparation Lemma in the same way as before to embed the required representing handles.

In the even-dimensional case $n = 2k$, we follow the proof of Wall's Theorem 5.6 in [**9**]. Let F: $Q \to Y$ be a bordism between k-connected maps f_0: $P_0 \to X \times 0$ and f_1: $P_1 \to X \times 1$. By regarding F as a map of triads, we see that we can do surgery rel ∂Q to make F: $Q \to Y$ k-connected. We can arrange that the only nonvanishing kernels form a short exact sequence

$$0 \to K_{k+1}(Q, \partial Q) \to K_k(\partial Q) \to K_k(Q) \to 0.$$

Furthermore, we can by performing trivial surgeries suppose that these kernels are free and based. It remains to show that $K_{k+1}(Q, \partial Q)$ is a kernel, for this we need a Poincaré analog of Wall's Lemma 5.7.

LEMMA 4.2. *Let* F: $(Q, P) \to (Y, X)$ *be a map of degree* 1, *where* (Q, P) *and* (Y, X) *are Poincaré duality pairs of formal dimension* $2k + 1 \geq 7$; *suppose F induces k-connected maps* $P \to X$ *and* $Q \to Y$ *and that* $K_k(N, M) = 0$ (*with coefficients* $\Lambda = Z[\pi_1(Y)]$). *Assume that the base of* $K_{k+1}(Q, P)$ *is free. Then* $K_{k+1}(Q, P)$ *is a subkernel in* $K_k(P)$.

PROOF (WALL). Since $F: P \to X$ is k-connected, it follows that $K_k(P)$ is generated as a Λ $(= Z[\pi_1(Y)])$-module by classes represented by maps of spheres. Let $x \in K_{k+1}(Q, P)$, and represent $\partial x \in K_k(P)$ as a sum of maps of spheres. We show these maps may be represented by framed immersions in P using the strong form of the Surgery Preparation Lemma. As in Wall we can find a map into Q of a $(k+1)$-sphere with discs removed, T, whose boundary spheres are mapped by the framed immersions in a smooth part of Q. Furthermore, F maps T into the 2-skeleton of (Y, X).

Thus for a basis $e_1, \ldots, e_r$ of $K_{k+1}(Q, P)$, we can find punctured spheres $T_1, \ldots, T_r$ representing the basis elements such that ∂T_i is a union of spheres representing e_i. If we wedge these maps together and apply the Surgery Preparation Lemma twice, once to the boundaries ∂T_i to get immersed spheres in P (the double points of the immersion are added 1-cells), then using the relative version to obtain a smooth region U of Q into which the T_i map with their boundaries framed immersed spheres in a smooth part of P. Then any elements $x, x' \in K_{k+1}(Q, P)$ can be represented by punctured spheres T, T' in U, and Wall's argument shows that the intersections $\lambda(\partial x, \partial x') = 0$ and $\mu(\partial x) = 0$.

END OF PROOF OF THEOREM 4.1. The rest of the bordism invariance of $\Theta(f)$ in the even-dimensional case follows exactly as in Wall. Theorem 4.1 is thus proved.

The following result is proved exactly as in the smooth case by working in a neighborhood of the (smooth!) 2-skeleton.

COROLLARY 4.3. *Let X be a Poincaré complex of formal dimension $n > 6$, and Θ an element of $L_n(\pi, (X), w)$. Then there exists a normal map $f: P \to X$ whose surgery obstruction $\Theta(f)$ is Θ.*

5. Low-dimensional cases. In this section we investigate how far our methods allow us to go if we wish to obtain results in lower dimensions than 7. Specifically, we show that one can prove the $(\pi\text{-}\pi)$ theorem in dimension 6 and that one can define surgery obstructions in dimension 5. We begin with the $(\pi\text{-}\pi)$ theorem.

THEOREM 5.1. *Let $f: (Q, P) \to (Y, X)$ be a degree-one map of Poincaré pairs of formal dimension 6. Suppose that the inclusion $Y \hookrightarrow X$ induces an isomorphism of fundamental groups. Then we can perform Poincaré surgery on P to make F a homotopy equivalence.*

PROOF. We can suppose surgery has already been done to make $f: P \to X$ 2-connected and $f: Q \to Y$ 3-connected. All four fundamental groups can thus be supposed isomorphic. Consider now the kernel $K_3(Q, P)$; we need to be able to kill this kernel. We can certainly do surgery on trivial 2-spheres in P so as to make $K_3(Q, P)$ free and based. Let $e_1, \ldots, e_l$ be such a preferred base. As ususl we get a map

$$A: \left(\bigvee_{i=1}^{l} D^3, \bigvee_{i=1}^{l} S^2\right) \to (Q, P)$$

representing the kernel. We would like to be able to apply the Surgery Preparation Lemma here but we do not have the full version; however, we do have the following lemma, which will be enough for our purposes.

LEMMA 5.2. *Suppose* $g\colon (Q, P) \to (Y, X)$ *is a degree-one normal map between Poincaré pairs of formal dimension* 6, *such that* $\pi_1(P) \simeq \pi_1(Q) \simeq \pi_1(Y) \simeq \pi_1(X)$. *Suppose that* (L, L_0) *is a finite CW-pair with* $\dim L = 3$, $\dim L_0 - 2$, *and that* $f\colon (L, L_0) \to (Q, P)$ *is a map such that the composition* $g \circ f | L_0$ *is homotopic to a map into the* 1*-skeleton of* P *and that this homotopy extends to a homotopy of pairs carrying* L *into the* 2*-skeleton of* Q. *Then there exists a CW-pair* (K, K_0) *where* $\dim(K - L) \leqslant 2$, $\dim(L - L_0) \leqslant 1$, *and* (Q, P) *splits as* $(N_1, \partial N_1) \cup_{N_0} (N_2, \partial N_2)$ *with the inclusion* $\partial N_1 - \operatorname{Int} N_0 \hookrightarrow N_1$ 1*-connected. Furthermore, there is a Poincaré embedding* $j\colon (K, K_0) \hookrightarrow (N_1, \partial N_1 - \operatorname{Int} N_0)$ *so that the following diagram commutes up to homotopy*:

$$\begin{array}{ccc} (L, L_0) & \xrightarrow{f} & (Q, P) \\ & \searrow \qquad\qquad \nearrow & \\ & (K, K_1) & \end{array}$$

Furthermore, $g \circ j | K_0$ *maps into the* 1*-skeleton of* X *and* $g \circ j | K$ *maps to the* 2*-skeleton of* P, *so that* $(N_1, \partial N_1)$ *has a smooth structure with* $\partial N_1 - \operatorname{Int} N_0$ *a smooth submanifold of* ∂N.

Before proving this lemma, we show how it is used to complete the proof of Theorem 5.1. Lemma 5.2 can be used to represent the generators of $K_3(Q, P)$ by maps that factor through a smooth manifold pair $(N_1, N_1 - \operatorname{Int} N_0)$ as in the previous cases. We have $(N_1, N_1 - \operatorname{Int} N_0)$ 1-connected which is enough to obtain the embeddings needed for the surgeries to kill $K_3(Q, P)$.

PROOF OF LEMMA 5.2. The method of proof of the first form of the Surgery Preparation Lemma works to give the complexes (K, K_0) and a Poincaré embedding $j\colon (K, K_0) \hookrightarrow (Q, P)$ such that $g \circ j(K_0)$ is contained in the 1-skeleton of P and $g \circ j(K)$ is contained in the 2-skeleton of Q. If $(N_1, \partial N_1) \cup_{M_0} (N_2, \partial N_2)$ is the associated splitting of (Q, P) mod boundary it remains to make the inclusion $\partial N_1 - \operatorname{Int} N_0 \hookrightarrow N_1$ 1-connected. We can perform handle exchanges on 1-handles using Levitt's results (which we quoted as Theorem 1.4) and add generators to $\pi_1(\partial N_1 - \operatorname{Int} N_0)$ so that it maps onto $\pi_1(P)$ and can modify N_1 so that $\pi_1(M) \simeq \pi_1(Q)$.

The smooth $(\pi\text{-}\dot{\pi})$ theorem then gives $(N_1, \partial N_1)$ a smooth structure with $(\partial N_1 - \operatorname{Int} N_0)$ a smooth codimension zero submanifold of ∂N_1 since K_0 has dimension 2 and 'smoothly' embeds in ∂N_1. Thus Lemma 5.2 is proved.

Our final result is the following

THEOREM 5.3. *Let* $f\colon P \to X$ *be a degree-one normal map of Poincaré complexes of formal dimension* 5; *then we can associate to* f *an element* $\Theta(f) \in L_5(\pi_1(X), w)$ *where* w *is the first Stiefel-Whitney class of* X. $\Theta(f)$ *depends only on the normal cobordism class of* f *and* $\Theta(f) = 0$ *if and only if one can do surgery on* f *to make it a homotopy equivalence.*

PROOF. We can make $f\colon P \to X$ 2-connected by surgery below the middle dimension; now using the boundary portion of Lemma 5.2 we can represent a basis for $K_2(P)$ by maps into a smooth region of P, and thus we can define a surgery obstruction as before. The invariance of $\Theta(f)$ follows as in the higher-dimensional cases.

REFERENCES

1. W. F. Browder, *Surgery on simply-connected manifolds*, Springer-Verlag, Berlin and New York, 1972.
2. J. P. E. Hodgson, *The Whitney technique for Poincaré complex embeddings*, Proc. Amer. Math. Soc. **35** (1972), 263–268.
3. ______, *General position in the Poincaré duality category*, Invent. Math. **24** (1974), 311–334.
4. L. Jones, *Patch spaces*, Ann. of Math. (2) **97** (1973), 306–343.
5. N. Levitt, *The structure of Poincaré duality spaces*, Topology **7** (1968), 369–388.
6. ______, *Applications of Poincaré duality spaces to the topology of manifolds*, Topology **11** (1972), 205–221.
7. F. Quinn, *Surgery on Poincaré and normal spaces*, Bull. Amer. Math. Soc. **78** (1972), 262–267.
8. M. Spivak, *Spaces satisfying Poincaré duality*, Topology **6** (1967), 77–102.
9. C. T. C. Wall, *Surgery on compact manifolds*, Academic Press, New York, 1971.
10. ______, *Classification problems in differential topology*. IV. *Thickenings*, Topology **5** (1966), 73–94.

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, ADELPHI UNIVERSITY, GARDEN CITY, NEW YORK 11530